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Monte Carlo Method for pricing of Bermuda type derivatives

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1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, \infty)}, P)$ be a filtered space with the usual condition, and $\{B_t\}_{t \in [0, \infty)}$ be a d -dimensional Brownian motion. Let $T > 0$, and let $\sigma : [0, T] \times \mathbf{R}^D \rightarrow \mathbf{R}^D \times \mathbf{R}^d$ and $b : [0, T] \times \mathbf{R}^D \rightarrow \mathbf{R}^D$ be continuous functions. For each $s \in [0, T]$ and $x \in \mathbf{R}^D$, let $X(t; s, x)$, $t \in [s, T]$ be a solution of the following SDE.

$$X(t; s, x) = x + \int_s^t \sigma(r, X(r; s, x)) dB_r + \int_s^t b(r, X(r; t, x)) dr, \quad t \in [s, T]. \quad (1)$$

We assume that the above SDE 1 has a path-wise unique solution for every $(s, x) \in [0, T] \times \mathbf{R}^D$.

Let $\tilde{\mathcal{S}}_s^t$, $0 \leq s \leq t \leq T$, be the set of \mathcal{F}_t -stopping times τ with $s \leq \tau \leq t$. Let $g : [0, T] \times \mathbf{R}^D \rightarrow \mathbf{R}$ be a continuous function with suitable conditions. Then, concerning the pricing of American derivatives, we are interested in computing the following value function,

$$u(s, x) = \sup\{E[g(\tau, X(\tau; s, x))]; \tau \in \tilde{\mathcal{S}}_s^T\}, (s, x) \in [0, T] \times \mathbf{R}^D.$$

There are several attempts to compute the value function u numerically. However, it seems that there are not so good method if D is not small. Let $N \geq 2$ and let T_n , $n = 0, 1, \dots, N$, be positive numbers such that $0 = T_0 < T_1 < \dots < T_N = T$. Let \mathcal{S}_n , $n = 0, 1, \dots, N$, be the set of \mathcal{F}_t -stopping times taking value in $\{T_n, T_{n+1}, \dots, T_N\}$. Concerning the pricing of Bermuda type derivatives, we are interested in computing the following value functions.

$$v_n(x) = \sup\{E[g(\tau, X(\tau; s, x))]; \tau \in \mathcal{S}_n\}, \quad n = 0, 1, \dots, N.$$

Let us define a probability measure $p_n(x, \cdot)$ over \mathbf{R}^D for each $n = 0, 1, \dots, N-1$, and $x \in \mathbf{R}^D$ by

$$p_n(x, A) = P(X(T_{n+1}; T_n, x) \in A), \quad \text{for a Borel set } A \text{ in } \mathbf{R}^D,$$

and define a operator P_n , $n = 0, 1, \dots, N-1$, by

$$P_n f(x) = \int_{\mathbf{R}^D} f(y) p_n(x, dy) = E[f(X(T_{n+1}; T_n, x))]$$

for a measurable function f on \mathbf{R}^D . Then v_n , $n = N, N-1, \dots, 0$, are given inductively by the following.

$$v_N(x) = g(T_N, x),$$

$$v_{n-1}(x) = (P_{n-1}v_n)(x) \vee g(T_{n-1}, x).$$

So the value function $v_0(x)$ is easily given mathematically. However, if D is not small, it is not easy to memorize a function on \mathbf{R}^D , and so it is not easy to compute $v_0(x)$.

Several people suggest a Monte-Carlo method to compute the value function. In this paper, we discuss the method given by [?]. We assume the following assumption (A).

(A1) D_n , $n = 0, 1, \dots, N-1$, are measurable sets in \mathbf{R}^N such that $(P_nv_{n+1})(x) \geq g(T_n, x)$ for any $x \in \mathbf{R}^D \setminus D_n$.

Remark 1 (1) $D_n = \mathbf{R}^D$ satisfies the assumption (A1).

(2) If $g(t, x) \geq 0$, for any $(t, x) \in [0, T] \times \mathbf{R}$, then $D_n = \{x \in \mathbf{R}^D; g(T_n, x) > 0\}$ satisfies the assumption (A1).

Now let $L_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\vec{X}_{n,\ell} = \{X_{n,\ell}(m)\}_{m=0}^N$, $\ell = 1, \dots, L_n$, $n = 0, 1, \dots, N-1$, are identically independent random vectors whose distribution is the same as the distribution of $\{X(T_m; 0, x)\}_{m=0}^N$. Let $K_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\psi_{n,k}$, $k = 1, \dots, K_n$, $n = 0, 1, \dots, N-1$, are functions on \mathbf{R}^D . Then we define functions H_n , $n = N, N-1, \dots, 1, 0$, on \mathbf{R}^D inductively by the following.

$$H_N(x) = 1.$$

When $\vec{H}_{n+1} = \{H_m\}_{m=n+1}^N$ are given we let

$$\sigma_{n,\ell} = \min\{m \geq n+1; H_m(X_{n,\ell}(m)) > 0\}, \quad \ell = 1, \dots, L_n.$$

Then we let $\{\tilde{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{\ell=1}^{L_n} |g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n,\ell}(\sigma_{n,\ell}))|^2 1_{D_n}(X_{n,\ell}(\sigma_{n,\ell})).$$

Finally we define H_n by

$$H_n(x) = \begin{cases} g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbf{R}^D \setminus D_n. \end{cases}$$

Then we let

$$\tilde{v}_0 = \frac{1}{L_0} \sum_{\ell=1}^{L_0} g(\sigma_{0,\ell}, X_{0,\ell}(\sigma_{0,\ell}))$$

and

$$\tilde{\sigma} = \min\{T_n; H_n(X(T_n; 0, x)) > 0\}.$$

We think that \tilde{v}_0 is an approximation of the value function $v_0(x)$ and the stopping time $\tilde{\sigma}$ as a candidate of the optimal stopping time.

2 Preliminary Results

Let $W_n = \mathbf{R}^{(N+1-n)D}$, $n = 0, 1, \dots, N$, and let $P_x^{(n)}$, $x \in \mathbf{R}^D$, be the distribution of $\{X(T_m; T_n, x)\}_{m=n}^N$ on W_n . Then $P_x^{(n)}$, $n = 0, 1, \dots, N$, $x \in \mathbf{R}^D$, is a Markov chain on \mathbf{R}^D .

For any measurable function h on \mathbf{R}^D and $n, m = 0, 1, \dots, N$ with $n \leq m$, let $\tau_m(\cdot; h) : W_n \rightarrow \{m, N\}$ by

$$\tau_m(w; h) = \begin{cases} m, & h(w(m)) > 0, \\ N, & h(w(m)) \leq 0. \end{cases}$$

Lemma 2 Let $h_n : \mathbf{R}^N \rightarrow \mathbf{R}$, $n = 0, 1, \dots, N$, be given, and assume that $h_n(x) \leq 0$, $x \in \mathbf{R}^N \setminus D_n$, and that $h_N(x) = 1$. Let $\sigma_n : W_n \rightarrow \{n, n+1, \dots, N\}$ be given by

$$\sigma_n(w) = \sigma_n(w; \{h_m\}_{m=n}^{N-1}) = \bigwedge_{m=n}^{N-1} \tau_m(w; h_m), \quad w \in W_n.$$

Moreover, let $u_n : \mathbf{R}^D \rightarrow \mathbf{R}$ be given by

$$u_n(x) = u_n(x; \{h_m\}_{m=n}^N) = E^{P_x^{(n)}}[g(T_{\sigma_n}, w(\sigma_n))], \quad x \in \mathbf{R}^D,$$

Then we have the following.

(1) $|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x)|P_n u_{n+1}(x) - (g(T_n, x) - h_n(x))|$
for any $n = 0, 1, \dots, N-1$, and $x \in \mathbf{R}^D$.

(2) $|u_n(x) - v_n(x)|$

$$\leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x) 1_{\{1\}}(\text{sgn}(P_n u_{n+1}(x) - g(T_n, x)) \text{sgn}(h_n(x))) |P_n u_{n+1}(x) - g(n, x)|.$$

Here

$$\text{sgn}(a) = \begin{cases} 1, & a > 0, \\ 0, & a = 0, \\ -1, & a < 0. \end{cases}$$

Proof. Note that $u_n(x) \leq v_n(x)$, for all $n = 0, 1, \dots, N-1$, and $x \in \mathbf{R}^D$. Let $\tilde{u}_n(x) = g(T_n, x) - h_n(x)$, $x \in \mathbf{R}^D$.

Let $n = 0, 1, \dots, N-1$, and $x \in \mathbf{R}^D$, and fix them for a while.

Case 1. Suppose that $h_n(x) > 0$.

Then we see that $x \in D_n$ and $g(T_n, x) > \tilde{u}_n(x)$. So we have

$$v_n(x) = g(T_n, x) + (P_n v_{n+1}(x) - g(T_n, x)) \vee 0 \leq g(T_n, x) + |P_n v_{n+1}(x) - \tilde{u}_n(x)|.$$

This implies

$$g(T_n, x) \geq v_n(x) - |P_n(v_{n+1} - u_{n+1})(x)| - |P_n u_{n+1}(x) - \tilde{u}_n(x)|.$$

Case 2. Suppose that $h_n(x) \leq 0$, and $x \in D_n$.

Then we see that $g(T_n, x) \leq \tilde{u}_n(x)$. So we see that

$$v_n(x) \leq P_n v_{n+1}(x) \vee \tilde{u}_n(x) \leq P_n u_{n+1}(x) + |P_n(v_{n+1} - u_{n+1})(x)| + |P_n u_{n+1}(x) - \tilde{u}_n(x)|.$$

Case 3. Suppose that $h_n(x) \leq 0$, and $x \in \mathbf{R}^D \setminus D_n$.
Then we see that $g(T_n, x) \leq (P_n v_{n+1})(x)$. So we have

$$v_n(x) = P_n v_{n+1}(x) \leq P_n u_{n+1}(x) + |P_n(u_{n+1} - v_{n+1})(x)|.$$

So we see that for any $n = 0, 1, \dots, N-1$,

$$\begin{aligned} u_n &= 1_{\{h_n > 0\}} g(T_n, \cdot) + 1_{\{h_n \leq 0\}} (P_n u_{n+1}) \\ &\geq 1_{\{h_n > 0\}} (v_n - |P_n(v_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|) \\ &\quad + 1_{\{h_n \leq 0\}} 1_{D_n} (v_n - |P_n(v_{n+1} - u_{n+1})| - |P_n u_{n+1} - \tilde{u}_n|) \\ &\quad + 1_{\{h_n \leq 0\}} 1_{\mathbf{R}^D \setminus D_n} (v_n - |P_n(v_{n+1} - u_{n+1})|). \end{aligned}$$

Thus we see that

$$0 \leq v_n - u_n \leq |P_n(v_{n+1} - u_{n+1})| + |1_{D_n} P_n u_{n+1} - \tilde{u}_n|.$$

This implies the assertion (1).

Now let us prove the assertion (2). Let ξ is a positive measurable function on \mathbf{R}^D .
Since $\tau_n(w; \xi h_n) = \tau_n(w; h_n)$, we see from the assertion (1) that

$$|u_n(x) - v_n(x)| \leq |P_n(u_{n+1} - v_{n+1})(x)| + 1_{D_n}(x) |P_n u_{n+1}(x) - g(T_n, x) + \xi(x) h_n(x)|.$$

Noting that

$$\inf\{a + tb; t > 0\} = 1_{\{1\}}(\operatorname{sgn}(a)\operatorname{sgn}(b))|a|, \quad a, b \in \mathbf{R},$$

we have the assertion (2).

This completes the proof. ■

Let ν_0 be a probability measure on \mathbf{R}^D and define probability measures ν_n , $n = 1, \dots, N$, inductively by

$$\nu_{n+1}(dx) = \int_{\mathbf{R}^D} p_n(y; dx) \nu_n(dy), \quad n = 0, 1, \dots, N-1.$$

Then we have the following as an easy consequence of Lemma 2.

Corollary 3 *Let h_n and u_n be the same as the previous lemma. Then we have the following.*

$$\begin{aligned} & \left(\int_{\mathbf{R}^D} |u_n(x) - v_n(x)|^2 \nu_n(dx) \right)^{1/2} \\ & \leq \left(\int_{\mathbf{R}^D} |u_{n+1} - v_{n+1}(x)|^2 \nu_{n+1}(dx) \right)^{1/2} + \int_{D_n} |P_n u_{n+1}(x) - (g(T_n, x) - h_n(x))|^2 \nu_n(dx) \end{aligned}$$

for any $n = 0, 1, \dots, N-1$.

3 Main Result

Let ν_0 be a probability measure over \mathbf{R}^D . Let $L_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\vec{X}_{n,\ell} = \{X_{n,\ell}(m)\}_{m=0}^N$, $\ell = 1, \dots, L_n$, $n = 0, 1, \dots, N-1$, are identically independent random vectors defined on the probability measure $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ whose distribution is $P_{\nu_0}^{(0)} = \int_{\mathbf{R}^D} P_x^{(0)} \nu_0(dx)$. Let $K_n \geq 1$, $n = 0, 1, \dots, N-1$, and $\psi_{n,k}$, $k = 1, \dots, K_n$, $n = 0, 1, \dots, N-1$, are functions on \mathbf{R}^D .

Then we define functions $H_n : \mathbf{R}^D \times \tilde{\Omega} \rightarrow \mathbf{R}$, $n = N, N-1, \dots, 1, 0$, on \mathbf{R}^D inductively by the following procedure.

$$H_N(x) = 1.$$

When $\vec{H}_{n+1} = \{H_m\}_{m=n+1}^N$, are given we let

$$\sigma_{n,\ell} = \min\{m \geq n+1; H_m(X_{n,\ell}(m)) > 0\}, \quad \ell = 1, \dots, L_n.$$

Then we let $\tilde{a}_n = \{\tilde{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$F_n(\{a_k\}_{k=1}^{K_n}) = \frac{1}{L_n} \sum_{\ell=1}^{L_n} |g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(X_{n,\ell}(n))|^2 1_{D_n}(X_{n,\ell}(n)).$$

Finally we define H_n by

$$H_n(x) = \begin{cases} g(T_n, x) - \sum_{k=1}^{K_n} \tilde{a}_{n,k} \psi_{n,k}(x), & x \in D_n \\ -1, & x \in \mathbf{R}^D \setminus D_n. \end{cases}$$

Let $U_n(x) = u_n(\cdot; \{H_m\}_{m=n}^N)(x)$. Here u_n is as in Lemma 2. Let $\bar{a}_n = \{\bar{a}_{n,k}\}_{k=1}^{K_n}$ be the minimizing point of the function

$$\bar{F}_n(\{a_k\}_{k=1}^{K_n}) = \int_{D_n} |(P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_n \psi_{n,k}(x)|^2 \nu_n(dx).$$

We assume the following.

(A2) $\psi_{n,k}$, $k = 1, \dots, K_n$, is linearly independent in $L^2(D_n; d\nu_n)$, $n = 0, 1, \dots, N-1$, where ν_n is the probability law of $w(n)$ under $P_{\nu_0}^{(0)}(dw)$.

(A3) $\int_{D_n} \psi_{n,k}(x)^4 \nu_n(dx) < \infty$ $k = 1, \dots, K_n$, $n = 0, 1, \dots, N-1$. and

$$\int_{\mathbf{R}^D} E^{P_x^{(0)}} [(\sum_{m=1}^N g(T_n, w(T_n)))^4] \nu_0(dx) < \infty, \quad n = 0, 1, \dots, N..$$

Then we have the following.

Theorem 4 (1) *There is a constant $C > 0$ such that*

$$E^{\tilde{P}}[1 \wedge (\sum_{k=1}^{K_n} |\tilde{a}_{n,k} - \bar{a}_{n,k}|^2)] \leq \frac{C}{L_n}.$$

$$\begin{aligned}
(2) & \left(\int_{\mathbf{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx) \right)^{1/2} \\
& \leq \left(\int_{\mathbf{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx) \right)^{1/2} + \left(\int_{\mathbf{R}^D} \left(\sum_{k=1}^{K_n} (\bar{a}_{n,k} - \bar{a}_{n,k}) \psi_{n,k}(x) \right)^2 \nu_n(dx) \right)^{1/2} \\
& \quad + \inf \left\{ \left(\int_{D_n} |(P_n U_{n+1})(x) - \sum_{k=1}^{K_n} a_k \psi_{n,k}(x)|^2 \nu_n(dx) \right)^{1/2}; a_k \in \mathbf{R}, k = 1, \dots, K_n \right\}
\end{aligned}$$

Proof. Let \mathcal{I}_n , $n = 0, 1, \dots, N-1$, be the σ -algebra generated by $\vec{X}_{n,\ell}$, $\ell = 1, \dots, L_n$, and let \mathcal{B}_n , $n = 0, 1, \dots, N-1$, be the σ -algebra generated by $\cup_{m=n}^{N-1} \mathcal{I}_m$. Inductively, we see that H_n is \mathcal{B}_n -measurable, $n = N-1, N-2, \dots, 0$. Also, we have

$$F_n(\{a_k\}_{k=1}^{K_n}) = \sum_{k,k'=1}^{K_n} C_{n,k,k'}^{(2)} a_k a_{k'} - 2 \sum_{k=1}^{K_n} c_{n,k}^{(1)} a_k + C_n^{(0)},$$

where

$$\begin{aligned}
C_{n,k,k'}^{(2)} &= \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k} \psi_{n,k'})(X_{n,\ell}) \\
c_{n,k}^{(1)} &= \frac{1}{L_n} \sum_{\ell=1}^{L_n} (1_{D_n} \psi_{n,k})(X_{n,\ell}) g(T_{\sigma_{n,\ell}}, X_{n,\ell}(\sigma_{n,\ell})).
\end{aligned}$$

Note that $\sigma_{n,\ell} = \sigma_{n+1}(\vec{X}_{n,\ell}(\cdot); \{H_m\}_{m=n+1}^N)$. Therefore we have

$$\bar{C}_{n,k,k'}^{(2)} = E^{\bar{P}}[C_{n,k,k'}^{(2)} | \mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) \psi_{n,k'}(x) \nu_n(dx), \quad (2)$$

and

$$\bar{c}_{n,k}^{(1)} = E^{\bar{P}}[c_{n,k}^{(1)} | \mathcal{B}_{n+1}] = \int_{D_n} \psi_{n,k}(x) (P_n U_{n+1})(x) \nu_n(dx). \quad (3)$$

Let $R_{n,k,k'}^{(2)} = C_{n,k,k'}^{(2)} - \bar{C}_{n,k,k'}^{(2)}$, and $r_{n,k}^{(1)} = c_{n,k}^{(1)} - \bar{c}_{n,k}^{(1)}$. Let $C_n^{(2)} = \{C_{n,k,k'}^{(2)}\}_{k,k'=1}^D$, $\bar{C}_n^{(2)} = \{\bar{C}_{n,k,k'}^{(2)}\}_{k,k'=1}^D$, and $R_n^{(2)} = \{R_{n,k,k'}^{(2)}\}_{k,k'=1}^D$ be $D \times D$ random matrices, and let $c_n^{(1)} = \{c_{n,k}^{(1)}\}_{k=1}^D$, $\bar{c}_n^{(1)} = \{\bar{c}_{n,k}^{(1)}\}_{k=1}^D$, and $r_n^{(1)} = \{r_{n,k}^{(1)}\}_{k=1}^D$ be D -dimensional random vectors. Then we see that

$$\bar{a}_n = C_n^{(2)-1} c_n^{(1)}, \quad \bar{a}_n = \bar{C}_n^{(2)-1} \bar{c}_n^{(1)}, \quad n = 0, \dots, N-1.$$

Also, we see that

$$\begin{aligned}
& E^{\bar{P}}[(R_{n,k,k'}^{(2)})^2] \\
&= \frac{1}{L_n} E[\text{Var}[1_{D_n}(X_{n,1}(n)) \psi_{n,k}(X_{n,1}(n)) \psi_{n,k'}(X_{n,1}(n)) | \mathcal{B}_{n+1}]] \\
&\leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 \psi_{n,k'}(x)^2 \nu_n(dx)
\end{aligned}$$

$$E^{\tilde{P}}[(r_{n,k}^{(1)})^2] = \frac{1}{L_n} E[Var[1_{D_n}(X_{n,1}(n))\psi_{n,k}(X_{n,1}(n))g(\sigma_{n,1}, X_{n,1}(\sigma_{n,1}))|\mathcal{B}_{n+1}]] \\ \leq \frac{1}{L_n} \int_{D_n} \psi_{n,k}(x)^2 E^{P_x^{(n)}}[g(T_{\sigma_{n+1}(w; \{H_m\}_{m=n+1}^N)})w(\sigma_{n+1}(w; \{H_m\}_{m=n+1}^N))]^2 \nu_n(dx).$$

If $\|\bar{C}_n^{(2)-1}R_n^{(2)}\| \leq 1/2$, we have

$$\|(\bar{C}_n^{(2)} + R_n^{(2)})^{-1}\bar{C}_n^{(2)-1}\| = \|(I + \bar{C}_n^{(2)-1}R_n^{(2)})^{-1} - I\|\bar{C}_n^{(2)-1} \leq 2\|\bar{C}_n^{(2)-1}\|\|R_n^{(2)}\|.$$

Here $\|\cdot\|$ is the operator norm of a matrix. So if $\|\bar{C}_n^{(2)-1}\|\|R_n^{(2)}\| \leq 1/2$ and $|\bar{c}_n^{(1)}| \leq 1$, we have

$$|\tilde{a}_n - \bar{a}_n| = |((\bar{C}_n^{(2)} + R_n^{(2)})^{-1} - \bar{C}_n^{(2)-1})(\bar{c}_n^{(1)} + r_n^{(1)}) + \bar{C}_n^{(2)-1}r_n^{(1)}| \\ \leq 2\|\bar{C}_n^{(2)-1}\|^2\|R_n^{(2)}\|(|\bar{c}_n^{(1)}| + 1)\|\bar{C}_n^{(2)-1}\|\|r_n^{(1)}\|$$

So we have

$$E^{\tilde{P}}[|\tilde{a}_n - \bar{a}_n|^2 \wedge 1] \\ \leq E^{\tilde{P}}[|\tilde{a}_n - \bar{a}_n|^2, \|\bar{C}_n^{(2)-1}\|\|R_n^{(2)}\| \leq 1/2, |\bar{c}_n^{(1)}| \leq 1] \\ + \tilde{P}(\|\bar{C}_n^{(2)-1}\|\|R_n^{(2)}\| > 1/2) + \tilde{P}(|\bar{c}_n^{(1)}| > 1) \\ \leq (4(|\bar{c}_n^{(1)}| + 1)^2\|\bar{C}_n^{(2)-1}\|^4 + 4\|\bar{C}_n^{(2)-1}\|^{-2})E^{\tilde{P}}[\|R_n^{(2)}\|^2] + (\|\bar{C}_n^{(2)-1}\|^2 + 1)E^{\tilde{P}}[|r_n^{(1)}|^2].$$

Also we have

$$|\bar{c}_n^{(1)}| \leq \left(\int_{D_n} \left(\sum_{k=1}^{K_n} \psi_{n,k}(x)^2\right) \nu_n(dx)\right)^{1/2} \left(\int_{\mathbf{R}^D} E^{P_x^{(0)}} \left[\left(\sum_{m=1}^N g(T_m, w(T_m))^2\right) \nu_0(dx)\right]\right)^{1/2},$$

$$E^{\tilde{P}}[\|R_n^{(2)}\|^2] \leq \frac{1}{L_n} \int_{D_n} \left(\sum_{k=1}^{K_n} \psi_{n,k}(x)^2\right)^2 \nu_n(dx),$$

and

$$E^{\tilde{P}}[|r_n^{(1)}|^2] \leq \frac{1}{L_n} \left(\int_{D_n} \left(\sum_{k=1}^{K_n} \psi_{n,k}(x)^2\right)^2 \nu_n(dx)\right)^{1/2} \left(\int_{\mathbf{R}^D} E^{P_x^{(0)}} \left[\left(\sum_{m=1}^N g(T_m, w(T_m))^4\right) \nu_0(dx)\right]\right)^{1/2}.$$

This implies the assertion (1).

The assertion (2) is an easy consequence of Lemma 2. ■

Let $V_n = \sum_{k=1}^{K_n} \mathbf{R}\psi_{n,k} \subset L^2(\mathbf{R}^D; d\nu_n)$, $n = 0, 1, \dots, N-1$. Then it is easy to see that U_n 's are determined by $\vec{X}_{n,\ell}$, $\ell = 1, \dots, L_n$, $n = 0, \dots, N$ and V_n 's and are independent of a choice of bases $\{\psi_{n,k}\}_{k=1}^{K_n}$. Let

$$d_n = \inf\left\{\left(\int_{\mathbf{R}^D} \left(\sum \psi_k(x)^2\right)^2 \nu_n(dx)\right)^{1/2}; \{\psi_k\}_{k=1}^{K_n} \text{ is a orhogonal basis of } V_n\right\},$$

and

$$c_0 = \left(\sum \int_{\mathbf{R}^D} E^{P_x^{(0)}} \left[\left(\sum_{m=1}^N g(T_m, w(T_m))^4\right) \nu_0(dx)\right]\right)^{1/4}.$$

Then we have the following from the proof of Theorem 4.

Corollary 5 $E[(\int_{\mathbf{R}^D} |U_n(x) - v_n(x)|^2 \nu_n(dx)) \wedge 1]^{1/2}$

$$\leq E[(\int_{\mathbf{R}^D} |U_{n+1}(x) - v_{n+1}(x)|^2 \nu_{n+1}(dx)) \wedge 1]^{1/2} + 4(L_n)^{-1/2} d_n(K_n^{1/2} c_0^{1/2} + 1)$$

$$+ E[\inf\{(\int_{D_n} |(P_n U_{n+1})(x) - \psi(x)|^2 \nu_n(dx)) \mid \psi \in V_n\} \wedge 1]^{1/2}.$$

References

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